Proving the obviously untrue

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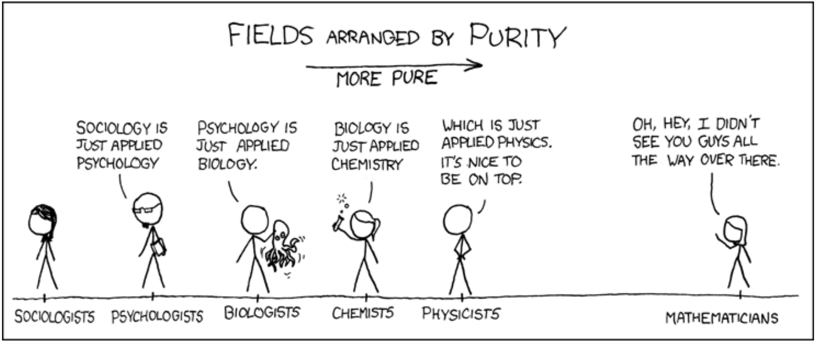
Outline

1. Introduction
   - Many a true word spoken in jest
   - Proving the obvious
2. Proving the not so obvious
3. Proving the obviously untrue
4. Seriously though

What is Mathematics?

What is mathematics, and is it important?



`http://xkcd.com/435/`

Outline

Introduction

Many a true word spoken in jest

Bluff your way in Maths, Robert Ainsley

"There are a lot of books about mathematics — usually very long ones with thousands of pages of small print without pictures, full of strings of odd-looking, apparently meaningless characters — like any university maths faculty. We can, however, classify the function of mathematics quite simply. Mathematics consists essentially of:

1. *proving the obvious;*
2. *proving the not so obvious; and*
3. *proving the obviously untrue."*

Outline

Proving the obvious.

Birthdays

Question?

What is the smallest number of people that one must have in one room to be sure that two of them share the same birthday?

Proving the obvious.

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Answer?

Many will say the answer is 366, but more pedantic observers will say that it is 367, one more than the number of days in a leap year.

Proving the obvious.
Birthdays

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The pigeon-hole principle

If $n > m$ pigeons are put into m pigeonholes, there's a hole with more than one pigeon.

Proving the not so obvious.

More Birthdays

Birthdays

So how many people must we gather on one room to have at least a 50% chance of having two people share the same birthday? (Neglect leap years this time)

Proving the not so obvious.

More Birthdays

Birthdays

So how many people must we gather on one room to have at least a 50% chance of having two people share the same birthday? (Neglect leap years this time)

Solution

If we have two people, the probability that they have different birthdays is given by the number of possible non-clashes, over the total number of days available.

$$\frac{364}{365}$$

Proving the not so obvious.

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Solution

Now if we add another person, we must also not clash with them, and there is one less available days to pick from.

$$\frac{364}{365} \times \frac{363}{365}$$

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Solution

We keep doing this, until the probability of a clash is just over 0.5, and it turns out we get to 23 people.

$$\frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{343}{365}$$

Proving the not so obvious.

Pigeon holing people again

Alexander Bogomolny

"At any given time in New York there live at least two people with the same number of hairs."

Proving the not so obvious.

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" I ran experiments with members of my family. My teenage son secured himself the highest marks sporting, in my estimate, about 900 hairs per square inch. Even assuming a pathological case of a 6 feet (two-sided) fellow 50 inch across, covered with hair head, neck, shoulders and so on down to the toes, ...

Proving the not so obvious.

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the fellow would have somewhere in the vicinity of 7,000,000 hairs which is probably a very gross over-estimate to start with. The Hammond's World Atlas I purchased some 15 years ago, estimates the population of the New York City between 7,500,000 and 9,000,000. The assertion therefore follows from the pigeonhole principle."

Proving the obviously untrue.

Why might this not work out?

The 50 Foot Woman

- Why might she have very sore feet and prefer a cup of tea instead?

Proving the obviously untrue.

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- Hint: how many times taller than a 5 foot woman is she?

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- Hint: how much bigger are the area of her feet than a 5 foot woman?
- Hint: how much heavier will she be than a 5 foot woman?

Proving the obviously untrue.

However, it is in the latter of Ainsley's categorisations that mathematics is often at its most interesting - in proving the obviously untrue.
These are the cautionary cases that explain the caution of mathematics, after all. Sometimes the obvious is wrong!

Proving the obviously untrue.

Roping the world

A long piece of rope

Imagine one has a rope long enough to straddle the whole way around the world (say on the equator).
Now, plant sticks to sit 1 m up from the surface of the earth.
How much extra rope to we need to go round the Earth 1 m above its surface?

Proving the obviously untrue.

Roping the world

A long piece of rope

- Of course, it's a clue that the radius of the Earth is not given. Let's call it R_E.
- The original rope has length $C_E = 2\pi R_E$,
- The new length is $C_N = 2\pi(R_E + 1)$.

So

$$C_N = 2\pi R_E + 2\pi \Rightarrow C_N = C_E + 2\pi$$

so only about another 6.28 m.

Doubling

Coins on a chess board

Question?

Here's an adapted version of an ancient problem. Suppose we place a single 10p piece on the first square of a chess board, then two on the next square, four on the next and so on. We double all the way through all 64 squares.

Doubling

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How tall (approximately) is the pile of coins on the last square?

Doubling

Coins on a chess board

We double each time, so the number of coins on the first eight squares are

$$1, 2, 4, 8, 16, 32, 64, 128.$$

Another way to write that is

$$2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7.$$

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So, we are going along 64 squares, and start at 0, so the number of coins on the last square will be

$$2^{63}.$$

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Any adjustments to make on your guess?

Doubling

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Now

$$2^{63} \approx 9,223,000,000,000,000,000 = 9.223 \times 10^{18}$$

and a ten pence piece is 1.85 mm thick, or 0.00185 m, so when we multiply these we get

$$17,063,000,000,000,000 \text{ m} = 1.7063 \times 10^{16} \text{ m}.$$

Wow. That seems like a lot. Just how big is that number as a distance?

Doubling

Coins on a chess board

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by the 19th square, the coins are higher than the radio mast on the Empire State building;

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Doubling

Examples in Computer Science

The Internet currently mainly runs on IPv4, which has, more or less 2^{32} addresses, so by our analogy more than 8 times as far as the Moon. But we are running out.

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Now a light-year is 9.4605284×10^{15} m. So in light-years this is

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The radius of the observable Universe is approximately 47,000,000,000 light years.

Doubling

Radioactive decay

Decay

What about halving? Suppose that you have a room of people, each with a coin. They all stand up.

Doubling
Radioactive decay

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Doubling
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Half life

So here, this ten seconds is the half life. This is exactly how radioactive decay works, with each person a particle or atom that might decay each time. So why the big deal?

Doubling

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Half lives

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Decay chains

Another complication is that very often, the material you decay into is itself radioactive.

Proving the obviously untrue.

1=2

Prove 1=2

Let $a = b$.

$$\Rightarrow a^2 = ab \Rightarrow a^2 + a^2 = a^2 + ab \Rightarrow 2a^2 = a^2 + ab$$

$$\Rightarrow 2a^2 - 2ab = a^2 + ab - 2ab \Rightarrow 2a^2 - 2ab = a^2 - ab$$

$$\Rightarrow 2(a^2 - ab) = 1(a^2 - ab)$$

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$$\Rightarrow 2a^2 - 2ab = a^2 + ab - 2ab \Rightarrow 2a^2 - 2ab = a^2 - ab$$

$$\Rightarrow 2(a^2 - ab) = 1(a^2 - ab)$$

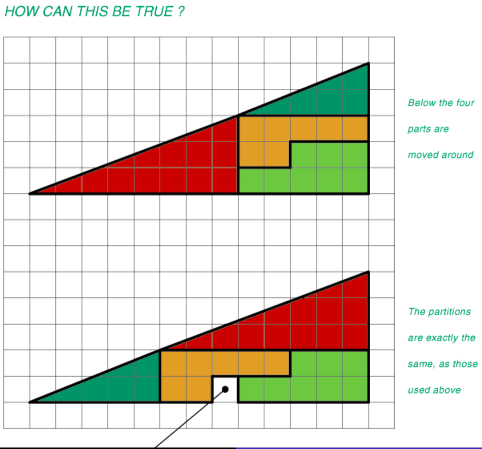
Now cancel $(a^2 - ab)$ both sides and we obtain

$$1 = 2$$

Proving the obviously untrue.

Back to areas...

Here is an example that arrived on my desk one day.



Proving the obviously untrue

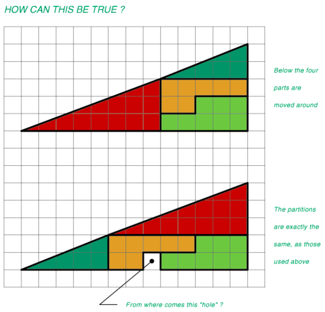
Back to areas...

Warning!

If you don't want to see the answer, look away now!

Proving the obviously untrue.

Back to areas...



Solution

- Look at the gradients of the red and green triangle upslope. If they were similar to the large triangle (which they should be) then they would be the same.
- The green triangle gradient is 2/5
- The red triangle gradient is 3/8
- The big triangle isn't actually a triangle.

Mobius strips

Take a strip of paper, give one end a half (180 degrees) twist and tape them together.

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- how many sides does it have?
- how many edges?

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- how many sides does it have?
- how many edges?
- what happens when you cut it in half?

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- or cut a third off the end?

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- how many edges?
- what happens when you cut it in half?
- or cut a third off the end?
- what use is it?

Proving the obviously untrue

The Spider and the Fly

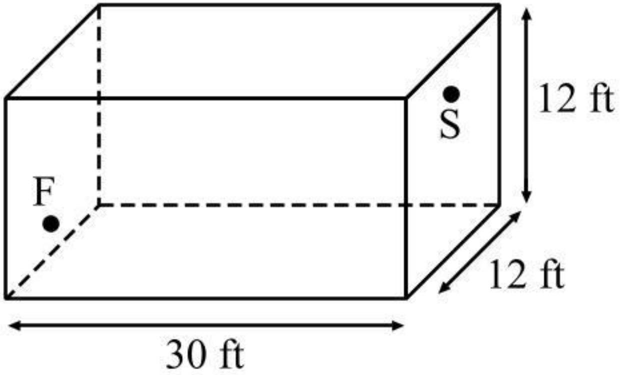
This fascinating example of the obviously untrue is accessible to almost all students, relying as it does on only the most elementary of mathematics.

We consider a room of dimensions 12 feet tall by 12 feet across by 30 feet long (since this is an old puzzle, or a very big room if we use metres).

The room has two occupants.

Proving the obviously untrue

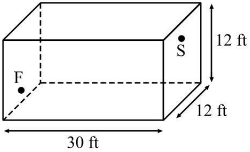
The Spider and the Fly



Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.

Proving the obviously untrue

The Spider and the Fly



Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.

Assumptions

1. the spider is very hungry, too hungry to spin silk;
2. therefore the spider stays on the walls at all times;
3. the spider has a degree in mathematics, or cognate subject.

Question?

What is the shortest distance the spider must travel to collect her prize? The *obvious* answer is 42, but it is wrong...

Proving the obviously untrue

The Spider and the Fly

Warning!

If you don't want to see the answer, look away now!

Proving the obviously untrue

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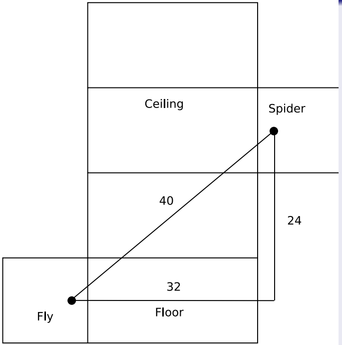


Solution

- As you can see, the correct answer travels over 5 out of the 6 walls!
- There are other possible unfoldings of course, some of which give the 42 answer we say before.
- Some students have complained this makes the whole thing a trick, since one can't unfold a real room the whole thing is impossible.

Proving the obviously untrue

The Linking Rings

Most of us will have seen the linking rings illusion; where a magician will link and unlink solid metal rings. This is clearly obviously untrue, or is it?

We shall consider another way of looking at this age old illusion.

The value of the obviously untrue

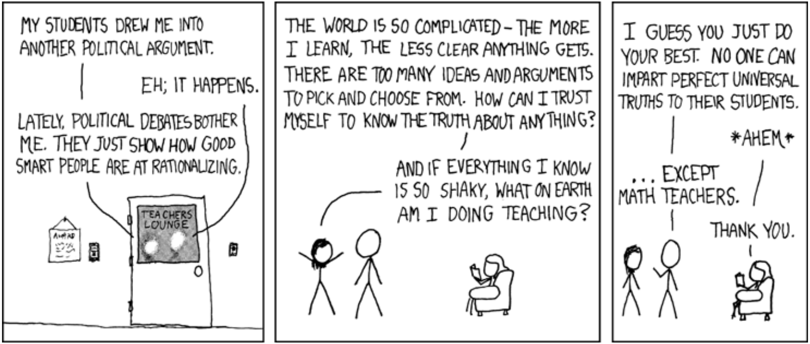
In fact these proofs are often the most informative. They give us deep insight into our assumptions and allow us to see the true structure of things.

Mathematics is about patterns, where they work and were they break. Patterns can be found in number, music, art, language, science, engineering ...

Summary

- The obvious is often false
- The obviously false is often true
- The examples that demonstrate these two crossovers in perception are often the most useful in demonstrating the real nature of things.

One for the Teachers



http://xkcd.com/263/