# Proving The Obviously Untrue

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# Outline

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## Bluff your way in Maths, Robert Ainsley

*"There are a lot of books about mathematics — usually very long ones with thousands of pages of small print without pictures, full of strings of odd-looking, apparently meaningless characters — like any university maths faculty. We can, however, classify the function of mathematics quite simply. Mathematics consists essentially of:*

1. *proving the obvious;*
2. *proving the not so obvious; and*
3. *proving the obviously untrue."*

# Outline

# Proving the obvious.

**Richard Stalford**

*"At university maths students actually spend their time proving 1+1=2."*

Mathematicians certainly have a reputation for stating the obvious, or for expending time worrying about things others happily accept as fact.

Let's look at some examples...

**Question?**

What is the smallest number of people that one must have in one room to be sure that two of them share the same birthday?

**Answer?**

Many will say the answer is 366, but more pedantic observers will say that it is 367, one more than the number of days in a leap year.

**The pigeon-hole principle**

If $n > m$ pigeons are put into $m$ pigeonholes, there's a hole with more than one pigeon.

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Cartographers have long known that four distinct colours are enough to colour a map in such a way that no two neighbouring "countries" are the same colour.

This problem defied proof by mathematicians for many years even though it appears to have crossed the desks of figures such as DeMorgan and Caylay.

Even the initial "proof" remained highly questioned.

An area which causes much confusion in under-graduate mathematicians is the idea of "proof by induction".
This powerful tool is normally employed to prove a statement that depends on a value $n$ for all natural numbers $n$.

The logic is often stated as follows:

### Proof by Induction

1. show true for $n = 1$;
2. assume true for $n$ and then show it is true for $n + 1$.

Then "by induction" is is true for all $n$ that are natural numbers.

## Example

Prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

## Example

1. show true for $n = 1$;

$$1^2 = \frac{1}{6} \times 1 \times 2 \times 3$$

Correct!

## Example

2 assume true for $n$ and then show it is true for $n+1$.

We have

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Add $(n+1)^2$ both sides

$$\Rightarrow 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

Let's work on the RHS

$$= (n+1)\left\{\frac{1}{6}n(2n+1) + (n+1)\right\}$$

## Example

$$= (n+1)\left\{\frac{1}{6}n(2n+1) + (n+1)\right\}$$

$$= (n+1)\left\{\frac{1}{6}\left(2n^2 + n + 6n + 6\right)\right\}$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

which is exactly what one would expect if $n$ was replaced by $n+1$ in the original assertion.

### The Problem

So what is the problem?
To many people, the second step involved assuming exactly what we were trying to prove in order to prove it. It seems a bit of a cheat.

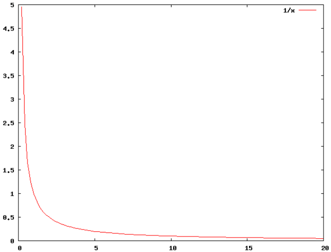
Here we begin to see the precision of language that is necessary to help mathematics. If we reword the logic it becomes clearer.

### Improved statement of Proof by induction

1. show true for $n = 1$;
2. assume true for a specific $n$ and then show it is true for $n + 1$.

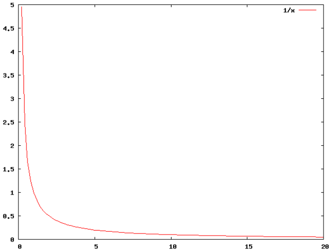
Hence by induction the statement is true for all natural numbers $n$.

Now this second statement is more reasonable, after all, we have an example from the first!



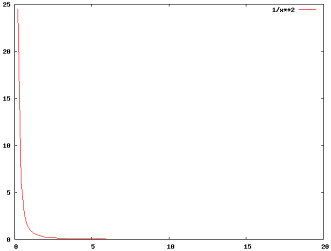

**Obvious?**

It's "obvious" that the area under the graph $y = 1/x$ from $x = 1$ to $x = \infty$ in infinite, since the graph never touches the $x$-axis.



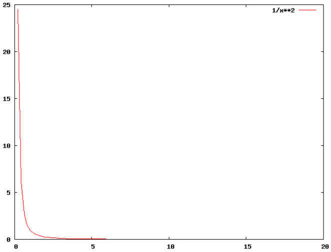

### And it is infinite

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{k\to\infty}\int_1^k \frac{1}{x}\,dx$$
$$= \lim_{k\to\infty}\left[\log_e x\right]_1^k$$
$$= \lim_{k\to\infty}(\log k - 0) = \infty$$




### Is it obvious?

It's also "obvious" that the area under the graph $y = 1/x^2$ from $x = 1$ to $x = \infty$ in infinite, since the graph never touches the $x$-axis. Although as the figure shows it gets incredibly close... but that shouldn't matter, should it?




**It is not true!**

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{k \to \infty} \int_1^k \frac{1}{x^2} dx$$

$$= \lim_{k \to \infty} \left[ -\frac{1}{x} \right]_1^k$$

$$= \lim_{k \to \infty} \left( -\frac{1}{k} - -\frac{1}{1} \right) = 1$$

### Results from the real line

This result stems from, and has obvious analogs in the analysis of the real number line. For example, it is "obvious" that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

and not only is it obvious, it is true! It is less obvious that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

has no such answer.

## OK, but so what?

These results are interesting, and tell us that something important happens when we start adding up infinitely many things that are becoming infinitely small. There is a delicate balancing act here. So much for the pure mathematics.

## Does it impact on the real world?

In fact, the ability to perform such integrals and get answers that are not infinite have important and interesting consequences in the mathematics that models the real world.

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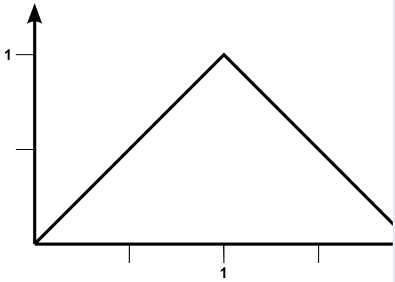


## Radiating Force

In classical physics forces are propogated by fields that (in general) spread out equally in all directions. If one considers the force "radiating" out from the source, then this force is spread increasingly thinly as the radius $r$ from the source increases. Consider the force spread out at a specific radius $r$, then it is distributed over a surface area $4\pi r^2$.

## The Inverse Square Law

As the original force is spread out over this sphere it follows that for these forces:

$$F = \frac{k}{r^2}$$

where $k$ is a (sometimes controversial) constant.

## Examples

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## Work Done is...

Consider the energy necessary for an object to be launched into space. We can use the more complete version of $E = Fs$, where $E$ is the work done, $F$ is the force and $s$ the distance travelled, since the force here is not a constant, we must use calculus:

$$E = \int_a^b F\, ds$$

gives the work done by a force in moving an object from $s = a$ to $s = b$.

## The mathematics

$$E = \int_{r_E}^{\infty} Fs \, ds = k \int_{r_E}^{\infty} \frac{1}{s^2} \, ds$$

Now because this is essentially $y = 1/x^2$, this integral is actually finite.

$$E = \frac{k}{R_E} = \frac{GM_E m}{R_E}$$

## Why it's important

If this were not so, it would be impossible for any object to be launched into space, or for any electron to be knocked out of orbit of an atom.

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This can be used to calculate the escape velocity for Earth, quite easily.

### By the way

$$\frac{1}{2}mv^2 = \frac{GM_E m}{R_E} \Rightarrow v = \sqrt{\frac{2GM_E}{R_E}}$$

### Schwarzchild Radius of a Black Hole $R_B$

Or to find the radius of a black hole in a similar way! Surprisingly, for a non rotating black hole, this is correct!

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So how many people must we gather on one room to have at least a 50% chance of having two people share the same birthday? (Neglect leap years this time)

### Solution

If we have two people, the probability that they have different birthdays is given by the number of possible non-clashes, over the total number of days available.

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**Alexander Bogomolny**

*"At any given time in New York there live at least two people with the same number of hairs."*

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*" I ran experiments with members of my family. My teenage son secured himself the highest marks sporting, in my estimate, about 900 hairs per square inch. Even assuming a pathological case of a 6 feet (two-sided) fellow 50 inch across, covered with hair head, neck, shoulders and so on down to the toes, ...*

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*the fellow would have somewhere in the vicinity of 7,000,000 hairs which is probably a very gross over-estimate to start with. The Hammond's World Atlas I purchased some 15 years ago, estimates the population of the New York City between 7,500,000 and 9,000,000. The assertion therefore follows from the pigeonhole principle."*

# Proving the obviously untrue.

However, it is in the latter of Ainsley's categorisations that mathematics is often at its most interesting - in proving the obviously untrue.

These are the cautionary cases that explain the caution of mathematics, after all. Sometimes the obvious is wrong!

### A long piece of rope

Imagine one has a rope long enough to straddle the whole way around the world (say on the equator).
Now, plant sticks to sit 1 m up from the surface of the earth.
How much extra rope to we need to go round the Earth 1 m above its surface?

### A long piece of rope

- Of course, it's a clue that the radius of the Earth is not given. Let's call it $R_E$.
- The original rope has length $C_E = 2\pi R_E$,
- The new length is $C_N = 2\pi(R_E + 1)$.

So

$$C_N = 2\pi R_E + 2\pi \Rightarrow C_N = C_E + 2\pi$$

so only about another 6.28 m.

**Prove 1=2**

Let $a = b$.

$$\Rightarrow a^2 = ab \Rightarrow a^2 + a^2 = a^2 + ab \Rightarrow 2a^2 = a^2 + ab$$

$$\Rightarrow 2a^2 - 2ab = a^2 + ab - 2ab \Rightarrow 2a^2 - 2ab = a^2 - ab$$

$$\Rightarrow 2(a^2 - ab) = 1(a^2 - ab)$$

Now cancel $(a^2 - ab)$ both sides and we obtain

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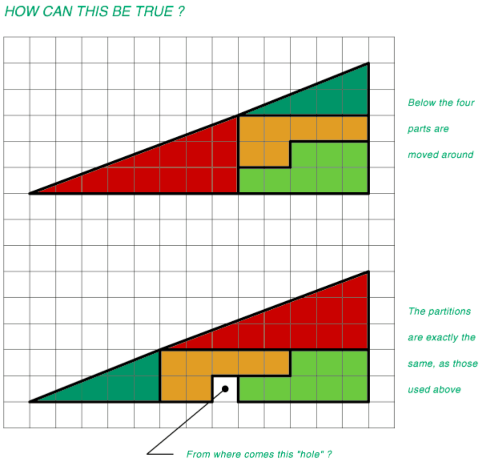
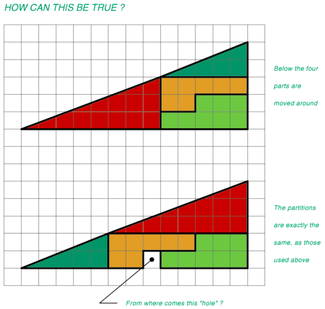
Here is an example that arrived on my desk one day.



**Warning!**

If you don't want to see the answer, look away now!




## Solution

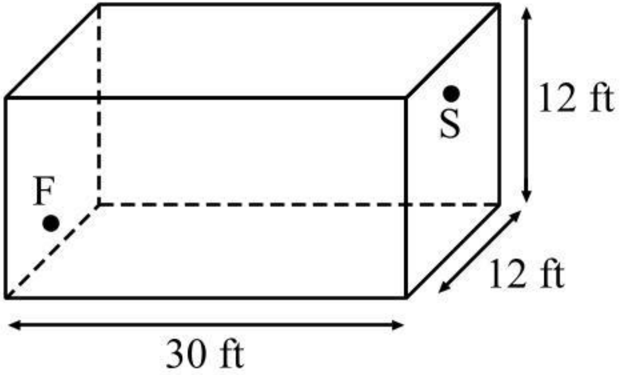
- Look at the gradients of the red and green triangle upslope. If they were similar to the large triangle (which they should be) then they would be the same.
- The green triangle gradient is $2/5$
- The red triangle gradient is $3/8$
- The big triangle isn't actually a triangle.

Most of us will have seen the linking rings illusion; where a magician will link and unlink solid metal rings. This is clearly obviously untrue, or is it?
We shall consider another way of looking at this age old illusion.

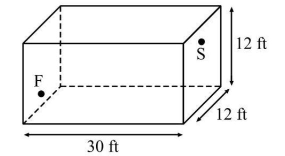
This fascinating example of the obviously untrue is accessible to almost all students, relying as it does on only the most elementary of mathematics.

We consider a room of dimensions 12 feet tall by 12 feet across by 30 feet long (since this is an old puzzle, or a very big room if we use metres).

The room has two occupants.




Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.



Spider centered on one wall, one foot from the ceiling.
Fly centered on opposite wall, one foot from the floor.

### Assumptions

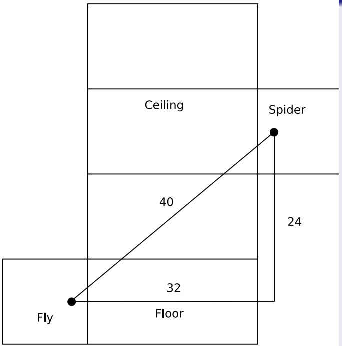
1. the spider is very hungry, too hungry to spin silk;
2. therefore the spider stays on the walls at all times;
3. the spider has a degree in mathematics, or cognate subject.

### Question?

What is the shortest distance the spider must travel to collect her prize? The *obvious* answer is 42, but it is wrong...

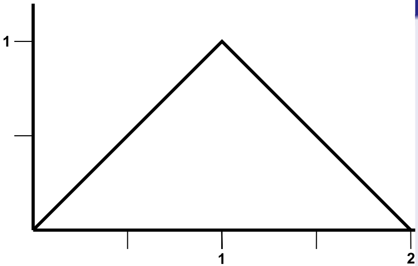
**Warning!**

If you don't want to see the answer, look away now!




## Solution

- As you can see, the correct answer travels over 5 out of the 6 walls!
- There are other possible unfoldings of course, some of which give the 42 answer we say before.
- Some students have complained this makes the whole thing a trick, since one can't unfold a real room the whole thing is impossible.

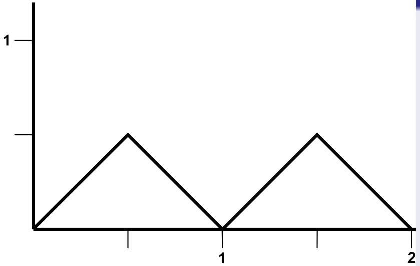



**Diagonals**

Consider the distance travelled following the line

$$d = 2\sqrt{1^2 + 1^2} = 2\sqrt{2}$$

Many people like to take short cuts through a grid system or a car park by going a bit more "diagonally".



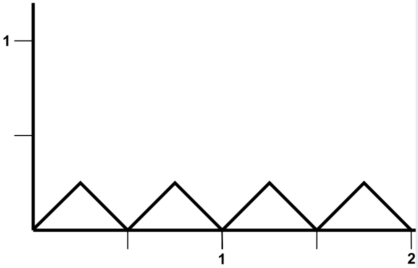

## Diagonals

Now look at the new distance

$$d = 4\sqrt{(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= 4\sqrt{\frac{1}{2}} = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$$

there is no reduction.




### Diagonals

No matter how many steps we go, there is no reduction. The areas get less and less under the curve, it gets closer and closer to the perfect straight line distance of 2, but it doesn't reach there smoothly. These sequence of function converges in both a pointwise and uniform manner to the straight line, and yet the lengths don't converge.

I leave this famous example till last since it is intimately related to the apparently innocuous but much debated "Axiom of Choice".

**Banach Tarski Theorem**

The theorem states that it is possible to take a solid sphere, cut it into a finite number of pieces, rearrange them using only rotations and translations, and re-assemble them into two identical copies of the original sphere.

In fact these proofs are often the most informative. They give us deep insight into our assumptions and allow us to see the true structure of things.

# In the real world

These cautionary notes may seem limited to the abtractness of pure mathematics, but in fact they are more widespread.

# Clinical Surrogates

In clinical research it is often expensive, or otherwise impractical to measure the real outcome of an experiment or trial.
For example, when a drug is used to counter the effects of a heart attack, it is hard to immediately measure the real outcome (long term mortality for example).
Sometimes surrogates are used instead. For example, it was taken as "obvious" that improvement observed in the 12 lead ECG would show lower mortality.

# Clinical Surrogates

The ECG was used as such a surrogate for trials on Lignocaine. The results showed a significant improvement in the 12-lead ECG after treatment.

Unfortunately it also killed more patients...

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# Summary

- The obvious is often false
- The obviously false is often true
- The examples that demonstrate these two crossovers in perception are often the most useful in demonstrating the real nature of things.