Necessary and Sufficient

A look at elegance, efficiency and completeness in Engineering and its Mathematics

Professor Colin Turner BSc Hons PhD FIET FIMA PFHEA CMath
Professor of Engineering Education

17th February 2016

Outline

Thank You

Aimee Helps Me Too

Fred Carroll (1945 - 2015)

Outline

Twitter and Web Materials

I'll be putting some web materials up for the lecture later today, and if you have questions on the way through I don't get to at the end, feel free to tweet them and I'll get to them later (if I know the answer).

Twitter ID	@ProfCTurner
Twitter hashtag	#nesssuff
Extra Credit	`http://www.piglets.org/inaugural` (BONUS CONTENT! But spoilers...)

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I KNOW WHO I AM.
I'M THE DUDE PLAYIN' THE DUDE, DISGUISED AS ANOTHER DUDE!

What is Engineering?

Engineering

Engineering is the application of mathematics, empirical evidence and scientific, economic, social, and practical knowledge in order to invent, innovate, design, build, maintain, research, and improve structures, machines, tools, systems, components, materials, and processes. The discipline of engineering is extremely broad, and encompasses a range of more specialized fields of engineering, each with a more specific emphasis on particular areas of applied science, technology and types of application. The term Engineering is derived from the Latin ingenium, meaning "cleverness" and ingeniare, meaning "to contrive, devise".

Definition of Engineering from Wikipedia in January 2016.

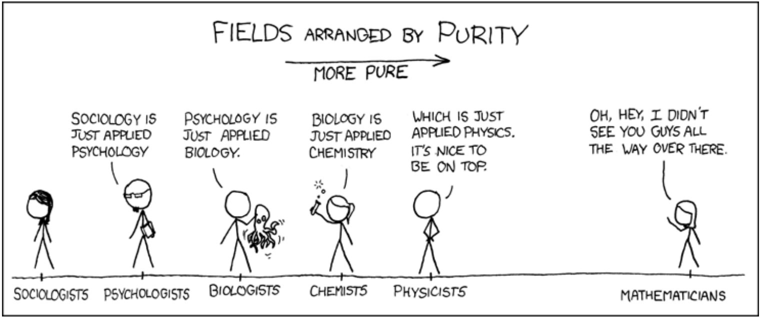
What is Mathematics?

Mathematics

(from Greek $\mu\acute{\alpha}\theta\nu\mu\alpha$ *mathema), "knowledge, study, learning") is the study of topics such as quantity (numbers), structure, space, and change. There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics.*

Definition of Mathematics from Wikipedia in January 2016.

What do we mean by Pure?



https://xkcd.com/435/

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Outline

The Treachery of Images

The Natural Numbers

Numbers arose to solve practical problems. And the same numbers arose in cultures around the Earth even if the numerals or system to represent them differed.

The so called Natural Numbers, or Counting Numbers, came first.

$$\mathbb{N} = 1, 2, 3, \ldots$$

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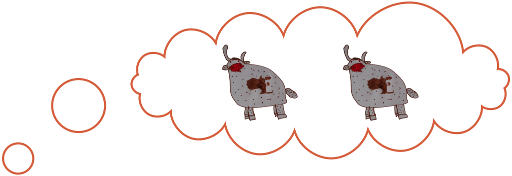

2 cows

As soon as there was “stuff” there was debt

Almost immediately cultures next start to consider some concept such as debt.

The Natural Numbers are no longer “sufficient” to fully represent this. We need to add some concept like negative numbers

$$\ldots -3, -2, -1$$

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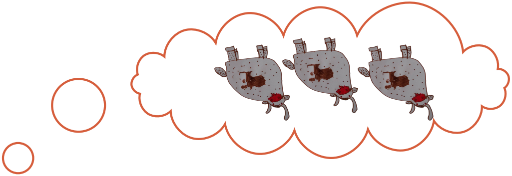

-3 cows

Multiplication and Division

Next we might need to multiply or divide by numbers. Multiplication is just a super charged addition, and division is its reverse. How might these have arisen?

Suddenly positive and negative “whole numbers” or “integers” are no longer sufficient. We need to add new numbers.

$$\ldots \frac{2}{3}, \frac{3}{5}, -\frac{3}{4} \ldots$$

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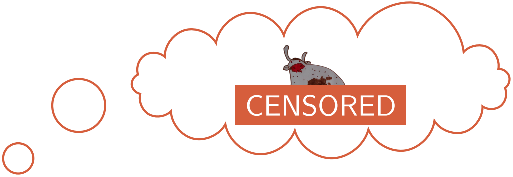

$\frac{2}{3}$ cows

Zero is a little weird

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Some cultures have found zero more or less concerning. For 20th / 21st Century people, that might seem bizarre.

But eventually we have the Integers.

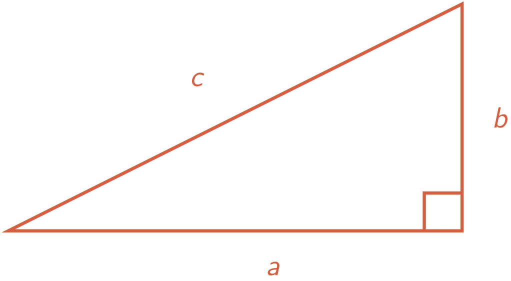
$$\mathbb{Z} = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots$$

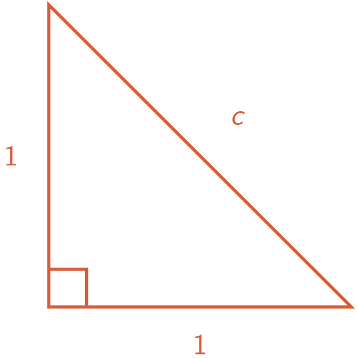
And along the way, a lot of fractions of these integers that we call the Rational Numbers ($\mathbb{Q}$ if you are interested).

This is enough. Both necessary and sufficient. Right?

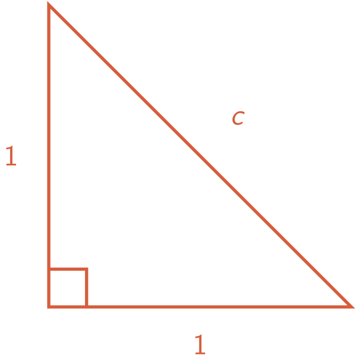
Most people will have met Pythagoras's Theorem while at School.

$$a^2 + b^2 = c^2$$





$$1^2 + 1^2 = c^2$$
$$2 = c^2 \Rightarrow c = \sqrt{2}$$



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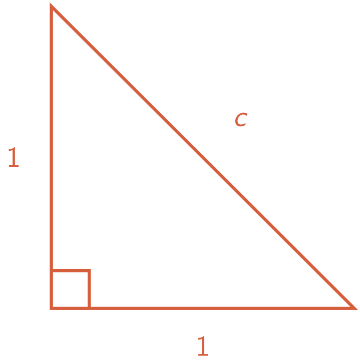
I have a truly marvellous proof that $\sqrt{2}$ is not rational (we say irrational), which this margin is too narrow to contain.



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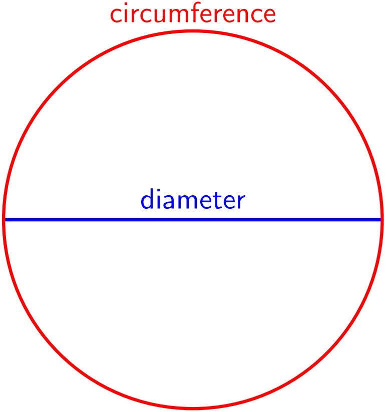
Really. I did try. See the website for a link.

Archimedes' Constant

One of the most important constants in reality is the ratio of the diameter of a circle to its circumference. It is the same for all circles.

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

But this number π is woven into the underlying reality. You find it *everywhere* in science and technology, where circles don't seem to be involved at all.



What about $\pi = \frac{22}{7}$?

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Remember “Jim’ll fix it”?

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What kind of a monster would ask a child to divide 7 into 22 for 50 decimal places, when it is only accurate as π for two?

$\frac{22}{7}$	$3.14285714286\ldots$
π	$3.14159265359\ldots$

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(Creative Commons
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$\frac{22}{7}$	3.14285714286 ...
π	3.14159265359 ...

Look closely at the calculation for $\frac{22}{7}$, even with this many decimal places you might notice something.

Is everything contained in π?

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But is it true?

Well, if it is true, it's not because it's irrational

$$0.101001000100001000001\ldots$$

is irrational, and doesn't even contain the number 2, for instance.

Euler's Number

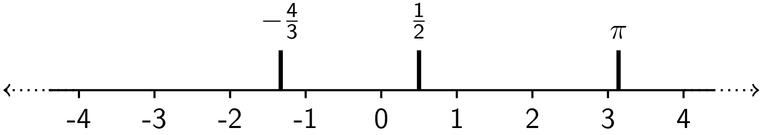
And π isn't the only ubiquitous constant that is irrational. This unusual number is also to be found in many equations.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \cdots$$

The value of e is approximately

$e = 2.71828182845904523536028747135266249775724709369995\ldots$

And so we come to the complete collection of all of that so far, the set of Real Numbers ($\mathbb{R}$). Sometimes we just refer to this as the Real Lines or the Continuum.



This has a really pleasing feeling of completeness. Every point on the line is a number, every number is a point on the line. Everything is necessary. And it is sufficient.

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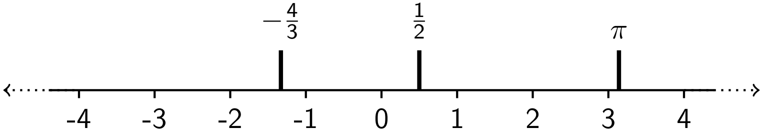


This has a really pleasing feeling of completeness. Every point on the line is a number, every number is a point on the line. Everything is necessary. And it is sufficient. Yes?

Square roots continue to deliver pain

What might the square root of 25, $\sqrt{-25}$ be? We are seeking a number, let's call it z so that $z \times z = z^2$ will be -25.

- it cannot be a positive real number, since a positive times a positive is positive (and so cannot possibly be -25);
- it cannot be a negative real number, since similarly a negative times a negative is positive;
- which leaves zero, but zero times zero is zero, and not -25.

So, it can't be a Real number. We could give up at this point, but we haven't done that before when we needed negative numbers, or irrational numbers to solve problems.

Square roots continue to deliver pain

It turns out we can split this problem up, a bit. We are allowed to split square roots up over multiplication.

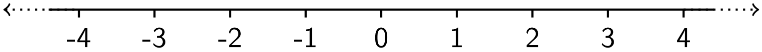
$$\sqrt{-25} = \sqrt{25 \times -1} = \sqrt{25} \times \sqrt{-1} = 5 \times \sqrt{-1}$$

This has the nice advantage of isolating the tricky bit, this minus sign, in just the -1. All square roots of negative numbers can be boiled down in this way. But we have the same problem with $\sqrt{-1}$, it can't be a Real number. What to do?

$$j = \sqrt{-1}$$

We make up a new number, j which isn't a Real number that has this value by definition. Now we can say $\sqrt{-25} = 5j$. Does this offend you?

Where should we put these?



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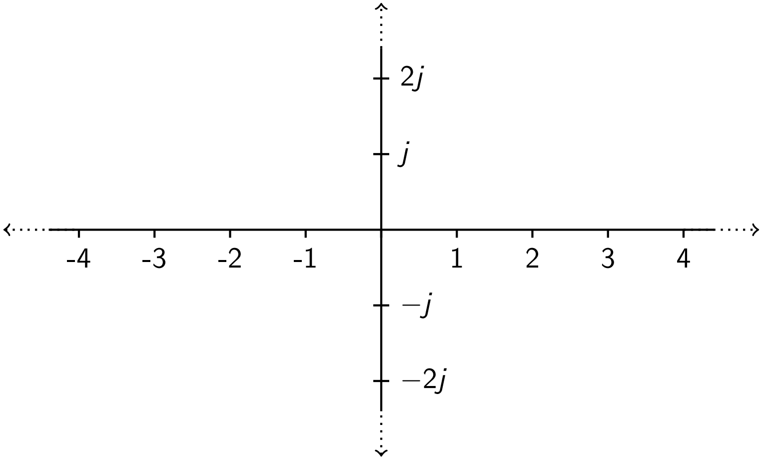

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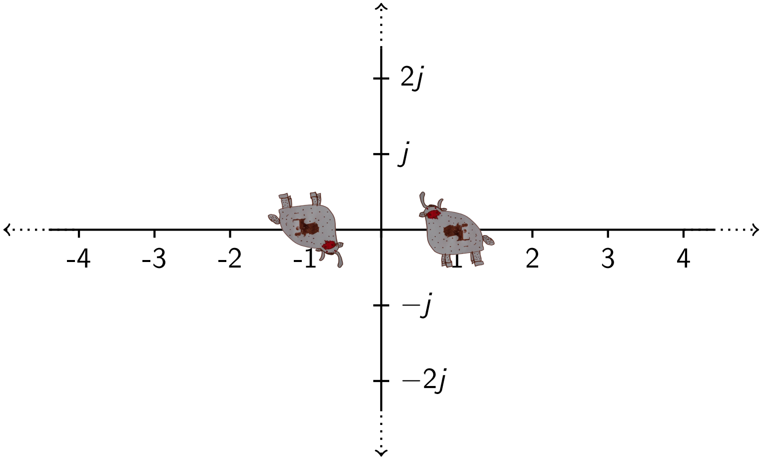

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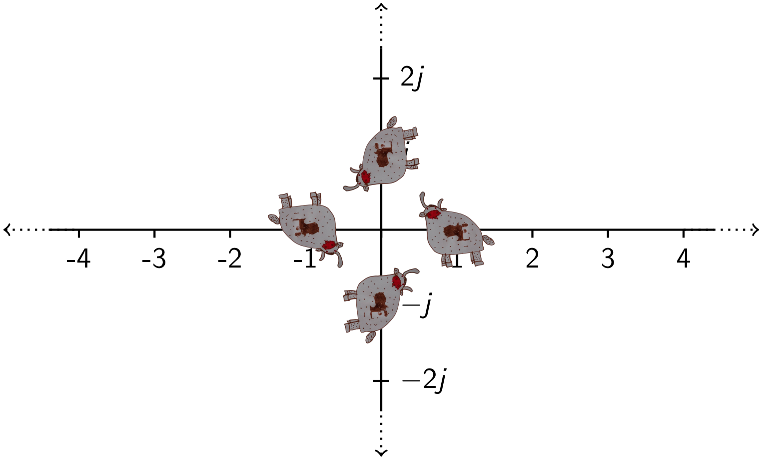

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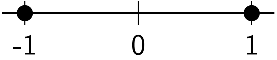
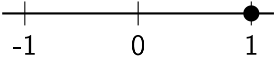
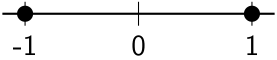
$\sqrt{1}$ (square root)



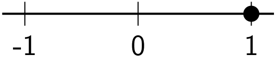
$\sqrt[3]{1}$ (cube root)

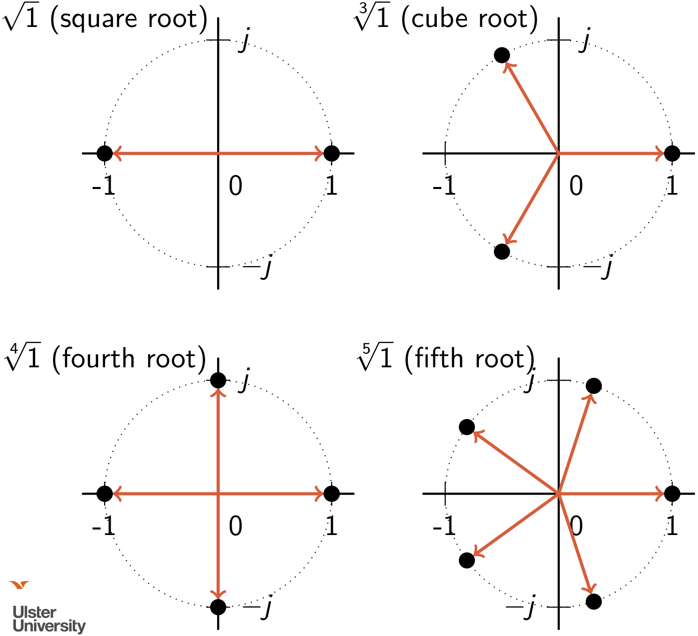


$\sqrt[4]{1}$ (fourth root)



$\sqrt[5]{1}$ (fifth root)





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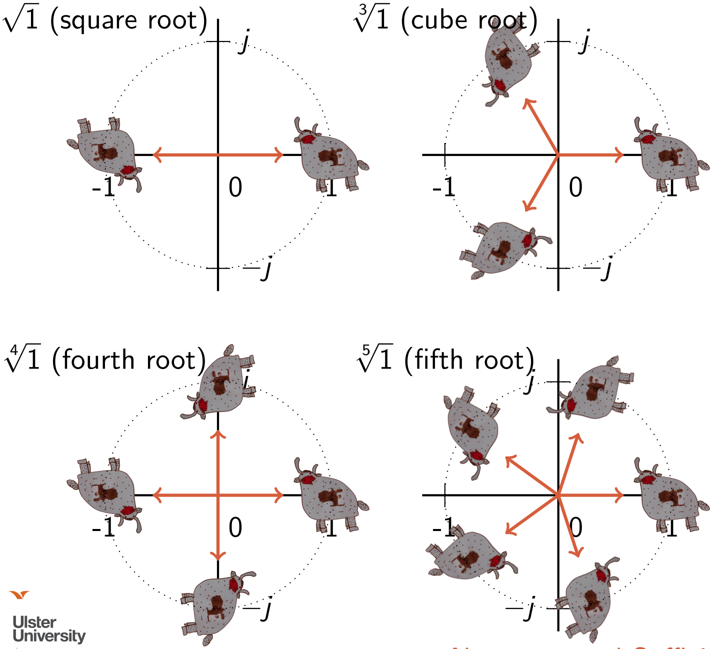

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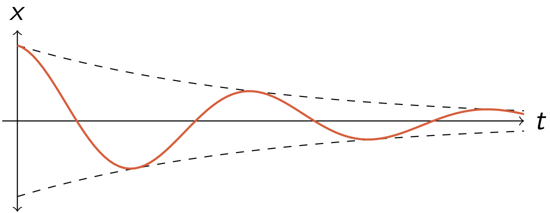
Necessary and Sufficient #nesssuff

Some applications

- Determining the relative size and position of ~~cows~~ circuit properties in an AC circuit, using phasors (not the ones you set to stun), which also use the "roots of unity" in for instance three phase mains electricity.
- Solving Differential Equations like this one

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = f(t)$$



Outline

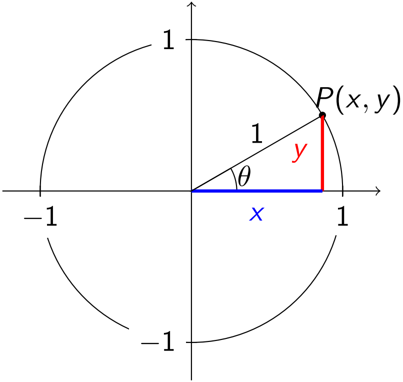
Is this the world's first exam question?

Rhind Mathematical Papyrus (c. 1680-1620 BC)

> *"If a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its seked?"*

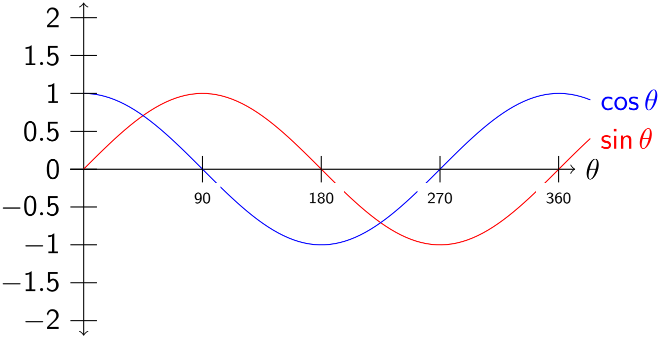
(For those interested, the seked seems to be the cotangent of the angle of the base of the pyramid to its face.)

Circular Functions



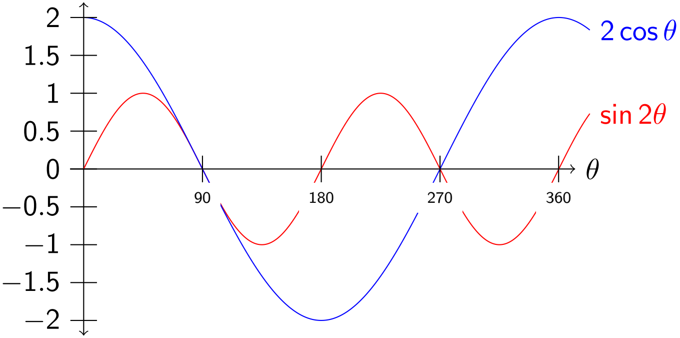
(θ is a Greek letter called theta that provides the th in many modern words).

Sine and Cosine



Amplitude and Frequency

The easiest way to manipulate these is to multiply the whole thing by a number to change the amplitude or to multiply the angle by a number to change the frequency.



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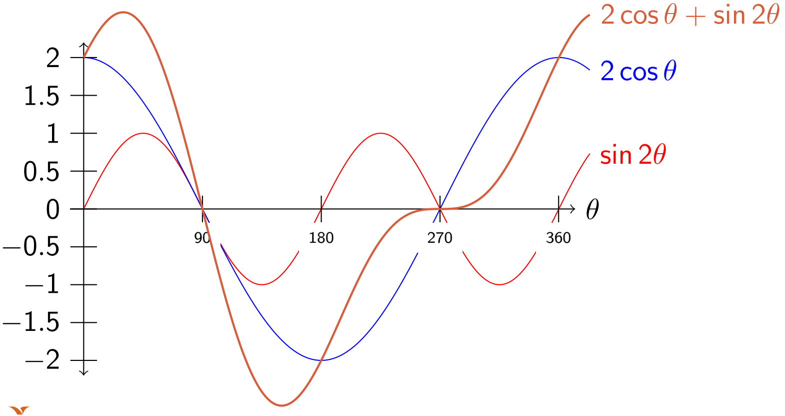
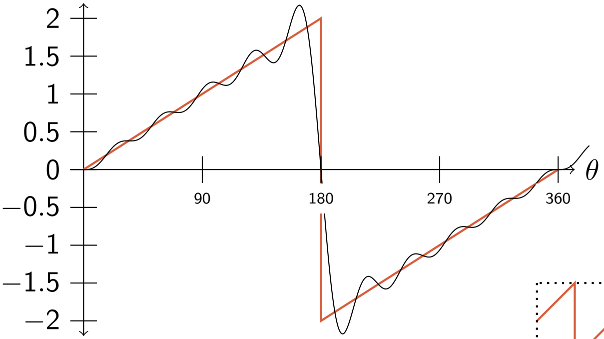


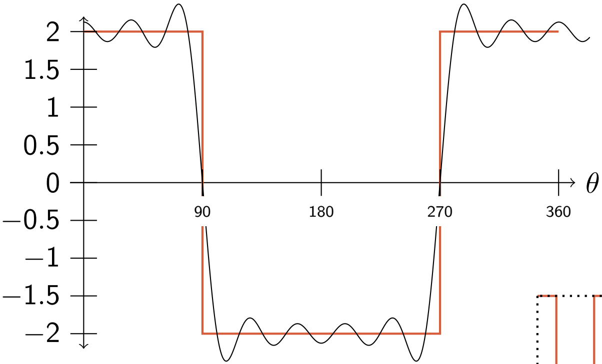
Combining Waves

$$\frac{4}{\pi}\left(\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \cdots\right)$$

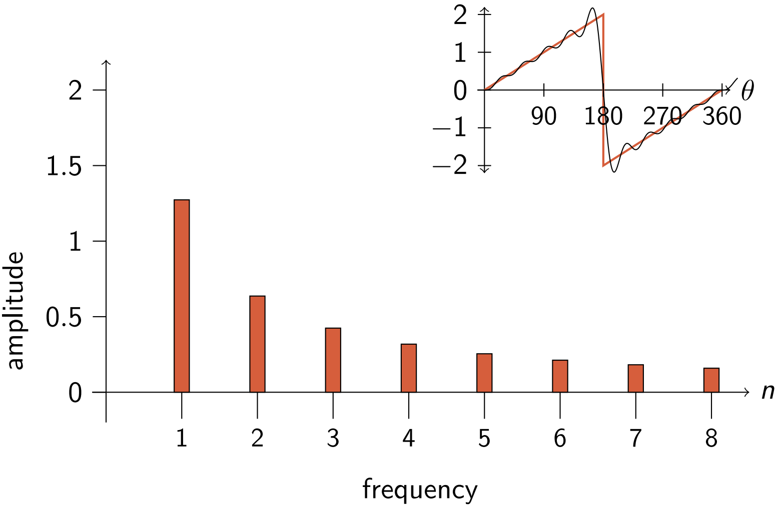


Combining Waves

$$\frac{8}{\pi}\left(\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \frac{\cos 7\theta}{7} + \cdots\right)$$



Frequency and Magnitude



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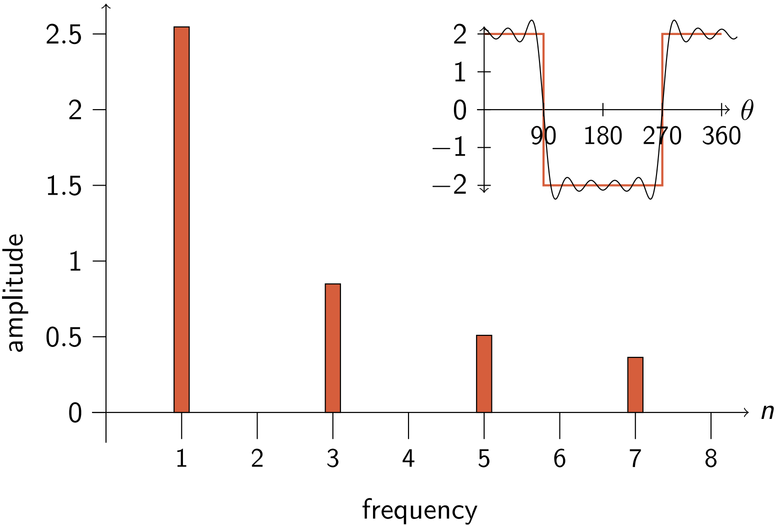

Some Applications

- We can often discard high frequency information and get an acceptable match to the input, allowing us to compress data;
- We can simulate, and interrogate information in many periodic signals, common to life, for instance the ECG;
- This just starts us on the path of using frequency rather than time as a way to analyse systems. It leads to other techniques such as
  - The Fourier Transform;
  - The Discrete Fourier Transform;

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi jkn/N} \quad k \in \mathbb{Z}$$

  - The Discrete Cosine Transform (used in compressed photos).
  - Laplace transforms, Z-transforms and many others.

Outline

The Avoidance of Routine

A word from a noted educationalist.

"A good teacher can never be fixed in a routine... each moment requires a sensitive mind that is constantly changing and constantly adapting. A teacher must never impose this student to fit his favourite pattern ..."

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Ulster University

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Bruce Lee

And another.

> *"You have got to think for yourself. You're all individuals. You're all different."*

Brian, Monty Python's Life of Brian

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I'm not

Disclaimer: This is not how I lecture...

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Obvious paradox detected...

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Student Feedback

"PowerPoint presentations explaining how to do a topic should be banned. I find it very hard to understand if the lecturer is just reading off the board. I prefer Professor Turner's way of teaching, through doing working examples etc. by writing on a page and projecting his page on to the screen. This way of teaching helps me understand better because at least I don't need to worry about the lecturer changing the slide and I'm left with a half written sentence/equation."

Stepping Stones

Chuck Norris, Bernie Sanders and Donald Trump in a boat

Chuck Norris, Bernie Sanders and Donald Trump go out on the lake in a boat. Suddenly, Sanders says, “I bet I can get to the shore of the lake without getting wet.” and proceeds to walk over, not getting wet. Chuck Norris decides to give it a try, and also walks across the lake. Trump tries, but immediately falls into the water. “Do you think we should have told him about the stepping stones” asks Sanders. Chuck Norris replies, “What stepping stones?”

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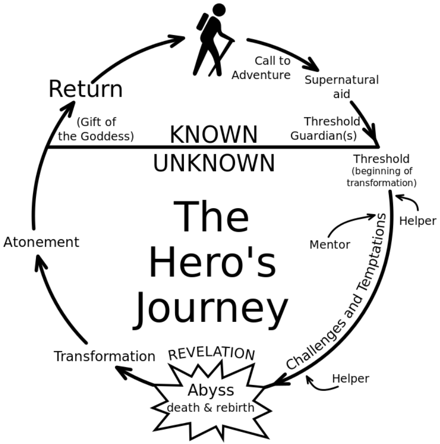
Maths is like this, there are many stones just under the water. Students can be taught to memorise the location of a few tens or hundreds of them. But in reality there is a simple structure which when understood allows a student to walk far without getting wet.

Gender, and expectation generally

> *"What precisely was it you wanted, madam?" she said. "It's only that I've left the class doing algebra, and they get restless when they've finished."*
>
> *"Algebra?" said Madam Frout, perforce staring at her own bosom, which no-one else had ever done. "But that's far too difficult for seven-year-olds!"*
>
> *"Yes, but I didn't tell them that and so far they haven't found out," said Susan.*

Terry Pratchett, Thief of Time

The Hero's Journey



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"We live on an island surrounded by a sea of ignorance. As our island of knowledge grows, so does the shore of our ignorance."

John Archibald Wheeler

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Reminder, extra content at
`http://www.piglets.org/inaugural`